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SEPARABLE SUBGROUPS HAVE BOUNDED PACKING

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Abstract. In this note, we prove that separable subgroups have bounded packing in ambient groups. The notion bounded packing was introduced by Hruska and Wise and in particular, our result answers positively a question of theirs, asking whether each subgroup of a virtually polycyclic group has the bounded packing property.

1. Introduction

Bounded packing was introduced for a subgroup of a countable group in Hruska-Wise [3]. Roughly speaking, this property gives a finite upper bound on the number of left cosets of the subgroup that are pairwise close in G. Precisely,

Definition. Let G be a countable group with a left invariant proper metric d. A subgroup H has *bounded packing* in G (with respect to d) if for each positive constant D, there is a natural number $N = N(G, H, D)$ such that, for any collection $\mathcal{C}$ of N left H-cosets in G, there exist at least two H-cosets $gH, g'H \in \mathcal{C}$ satisfying $d(gH, g'H) > D$.

Remark. Bounded packing of a subgroup is independent of the choice of the left invariant proper metric d. Equivalently, bounded packing says that for each positive constant D, every collection of left H-cosets in G with pairwise distance at most D has a uniform bound $N = N(G, H, D)$ on their cardinality.

This note aims to give a proof of the following.

Theorem. *If H is a separable subgroup of a countable group G, then H has bounded packing in G.*

A subgroup H of a group G is *separable* if H is an intersection of finite index subgroups of G. A group is called *subgroup separable* or *LERF* if every finitely generated subgroup is separable. For example, Hall showed that free groups are LERF in [1]. It follows from a theorem of Mal'cev [4] that polycyclic (and in particular finitely generated nilpotent) groups are LERF. A group is called *slender* if every subgroup is finitely generated. Polycyclic groups are also slender by a result of Hirsch [2]. Therefore, we have the following corollary, which gives a positive answer to [3, Conjecture 2.14].

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Corollary. *Let P be virtually polycyclic. Then each subgroup of P has bounded packing in P.*

Remark. In [5], Jordan Sahattchieve obtained a special case of this Corollary using different methods: any subgroup of (Hirsch) length 1 of a polycyclic group has bounded packing.

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2. Proof of the Theorem

We define the norm $|g|_d$ of an element $g \in G$ as the distance $d(1, g)$.

Proof of the Theorem. By the definition of bounded packing, it suffices to show, for each positive constant D, that there is a uniform bound on the cardinality of every collection of left H-cosets in G with pairwise distance at most D.

Given such a collection $\mathcal{A}$ satisfying $d(gH, g'H) < D$ for any gH, $g'H \in \mathcal{A}$. Without loss of generality, we can assume H belongs to $\mathcal{A}$, up to a translation of $\mathcal{A}$ by an appropriate element of G. Since $d(H, gH) < D$ for each $gH \in \mathcal{A}$, there exists an element h in H such that $d(1, hgH) < D$. Hence we conclude that the collection $\mathcal{A} \setminus \{H\}$ lies in the finite union of double cosets HgH with $|g|_d < D$ and $g \in G \setminus H$.

Since d is a left invariant proper metric on G, the set $F = \{g \in G \setminus H : |g|_d < D\}$ is finite. Since H is separable in G, we can take a finite index subgroup K of G such that $H < K$ and $F \subset G \setminus K$.

We claim that no two different left H-cosets of $\mathcal{A}$ lie in the same left K-coset. By way of contradiction, we suppose that there are two H-cosets gkH, $gk'H \in \mathcal{A}$ in the same coset gK such that $d(gkH, gk'H) < D$. By a similar argument as above, we get that $k^{-1}k'H$ belongs to a double coset Hg_0H with $|g_0|_d < D$. Moreover, we note that $g_0 \in F$. Since we have $k^{-1}k'H = hg_0H$ for some $h \in H$, it is easy to see that g_0 belongs to K. But by the choice of K, we know that g_0 belongs to $G \setminus K$. This is a contradiction. Our claim is proved.

Since K is of finite index in G, the cardinality of each $\mathcal{A}$ is upper bounded by $[G : K]$. Thus for each D, we have obtained a uniform bound on every $\mathcal{A}$. Hence H has bounded packing in G. □

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